Hedging and the Competitive Firm under Ambiguous Price and Background Risk *

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This paper examines the optimal production and hedging decisions of the competitive firm that possesses smooth ambiguity preferences and faces ambiguous price and background risk. The separation theorem holds in that the firm’s optimal output level depends neither on the firm’s attitudes towards ambiguity nor on the incidents to the underlying ambiguity. The full-hedging theorem holds only when the background risk is unambiguous. When the price risk is unambiguous, futures contracts become redundant but option contracts are not due to the prevalence of ambiguity driven by the volatility of the price risk, thereby making the latter effective in hedging against the ambiguous background risk. In the general case that both the price risk and the background risk are ambiguous, examples are constructed to show that options play a role as a hedging instrument over and above that of futures.

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HEDGING AND THE COMPETITIVE FIRM UNDER AMBIGUOUS PRICE AND BACKGROUND RISK

This paper examines the optimal production and hedging decisions of the competitive firm that possesses smooth ambiguity preferences and faces ambiguous price and background risk. The separation theorem holds in that the firm’s optimal output level depends neither on the firm’s attitudes towards ambiguity nor on the incidents to the underlying ambiguity. The full-hedging theorem holds only when the background risk is unambiguous. When the price risk is unambiguous, futures contracts become redundant but option contracts are not due to the prevalence of ambiguity driven by the volatility of the price risk, thereby making the latter effective in hedging against the ambiguous background risk. In the general case that both the price risk and the background risk are ambiguous, examples are constructed to show that options play a role as a hedging instrument over and above that of futures.

INTRODUCTION

Since the seminal work of Sandmo (1971), the theory of the competitive firm under price uncertainty has been extensively studied. One important strand of this literature is on the behavior of the firm when a futures market exists (Danthine, 1978; Feder et al., 1980; Holthausen, 1979), from which two notable results emanate. First, the separation theorem states that the firm’s production decision depends neither on the firm’s preferences nor on the underlying price distribution. Second, the full-hedging theorem asserts that the firm should completely eliminate its exposure to the price risk by adopting a full-hedge if the futures market is unbiased.\(^1\) A corollary of the full-hedging theorem is that no other hedging instruments, options in particular, play a role that is over and above that of futures for hedging purposes (Battermann et al., 2000).\(^2\)

\(^1\)The full-hedging theorem is analogous to a well-known result in the insurance literature that a risk-averse individual fully insures at an actuarially fair price (Mossin, 1968).

\(^2\)Lapan et al. (1991) show that options are used by the competitive firm only when the futures price and/or option premiums are perceived by the firm as biased. Options as such appear to be a speculative device rather than a hedging instrument.
hedging theorems in general, and the hedging role of options in particular, under the premise that the competitive firm of Sandmo (1971) is unable to unambiguously assign probability distributions that uniquely describe the price risk and other sources of risk that are aggregated into additive background risk.\(^3\) In other words, the firm faces ambiguity, or uncertainty in the sense of Knight (1921), about the price and background risk.\(^4\) Ambiguity averse preferences are supported by convincing evidence from many experiments (Chow and Sarin, 2001; Einhorn and Hogarth, 1986; Sarin and Weber, 1993) and surveys (Chesson and Viscusi, 2003; Viscusi and Chesson, 1999), which document that individuals prefer gambles with known rather than unknown probabilities.\(^5\)

Klibanoff et al. (2005) have recently developed a powerful decision criterion known as “smooth ambiguity aversion” that is compatible with ambiguity averse preferences under uncertainty (hereafter referred to as the KMM model). The KMM model features the recursive structure that is far more tractable in comparison to other models of ambiguity such as the pioneering maxmin expected utility (or multiple-prior) model of Gilboa and Schmeidler (1989).\(^6\) Specifically, the KMM model represents ambiguity by a second-order probability distribution that captures the firm’s uncertainty about which of the subjective beliefs govern the price and background risk. The KMM model then measures the firm’s expected utility under ambiguity by taking the (second-order) expectation of a concave transformation of the (first-order) expected utility of profit conditional on each plausible subjective joint distribution of the price and background risk.\(^7\) This recursive structure

\(^3\)Examples of background risk abound. Some of the initial wealth of the competitive firm may be held in risky assets and thus creates an additional source of uncertainty (Chavas, 1985). The fixed cost of the competitive firm may also be random because the firm’s physical assets can be ruined by natural disasters or fire (Wong, 1996). See also Osaki and Schlesinger (2014).

\(^4\)Knight (1921) points out that ambiguity is fundamentally different from risk. Specifically, risk relates to objective uncertainty, where outcome probabilities are known or can be estimated with confidence. In contrast, ambiguity relates to subjective uncertainty, where outcome probabilities are unknown, and decision makers are not sure which estimated models are correct.

\(^5\)Dated back to the Ellsberg’s (1961) paradox, ambiguity has been alluded to the violation of the independence axiom, which is responsible for the decision criterion being linear in the outcome probabilities. See Epstein and Schneider (2010) for a review of the ambiguity literature.


\(^7\)Skiadas (2013) shows that smooth ambiguity preferences can be approximated by preferences admitting an expected utility representation in continuous-time or high-frequency models under Brownian or Poisson uncertainty.
creates a crisp separation between ambiguity and ambiguity aversion, i.e., between beliefs and tastes, which allows these two attributes to be studied independently. Another nice feature of the KMM model is that the conventional techniques in the decision making under uncertainty are applicable in the context of ambiguity (Alary et al., 2013; Gollier, 2011; Snow, 2010, 2011; Taboga, 2005; Treich, 2010; Wong, 2015).

Within the KMM model, this paper shows that the separation theorem holds in that the firm’s optimal output level depends neither on the firm’s attitudes towards ambiguity nor on the incidents to the underlying ambiguity. If the firm is risk neutral, it regards the price and background risk as unambiguous whenever they have constant conditional means. Given that the firm can trade fairly priced futures and option contracts for hedging purposes, the full-hedging theorem holds only when the background risk is unambiguous. When the price risk is unambiguous, the unbiased futures contracts become redundant and play no hedging role. The option contracts, however, have conditional expected payoffs that entail ambiguity driven by the volatility of the price risk. Hence, the option contracts serve as a useful hedging instrument to hedge against the ambiguous background risk. In the general case that both the price risk and the background risk are ambiguous, examples are constructed to show that the firm indeed includes the option contracts in its optimal hedge position.

This paper is related to the burgeoning literature on the hedging role of options. Moschini and Lapan (1992) and Wong (2003a) show that export flexibility leads to an ex-ante profit function that is convex in prices. This induced convexity makes options a useful hedging instrument. Moschini and Lapan (1995), Brown and Toft (2002), Wong (2003b), Lien and Wong (2004), and Korn (2009) show that firms facing both hedgeable and non-hedgeable risks optimally use options for hedging purposes. The hedging demand for options in this case arises from the fact that the two sources of uncertainty interact in a multiplicative manner, which affects the curvature of profit functions. Frechette (2001) demonstrates the value of options in a hedge portfolio when there are transaction costs, even though markets themselves may be unbiased. Futures and options are shown to be highly substi-
tutable and the optimal mix of them are rarely one-sided. Lien and Wong (2002) justify the hedging role of options with multiple delivery specifications in futures markets. The presence of delivery risk creates a truncation of the price distribution, thereby calling for the use of options as a hedging instrument. Chang and Wong (2003) theoretically derive and empirically document the merits of using currency options for cross-hedging purposes, which are due to a triangular parity condition among related spot exchange rates. Wong and Xu (2006) and Adam-Müller and Panaretou (2009) show that the presence of liquidity risk truncates a firm’s payoff profile, thereby making options particularly suitable for such a hedging need. Benninga and Oosterhof (2004) show that options are used by firms whose private state prices differ from the market state prices. Wong (2012) shows the hedging role of options under state-dependent preferences.

The rest of this paper is organized as follows. The next section delineates the KMM model of the competitive firm facing ambiguous price and background risk. The subsequent section characterizes the firm’s optimal production decision. The following section derives the firm’s optimal hedging decision. The penultimate section constructs parametric examples for comparative static analysis. The final section concludes.

THE MODEL

Consider the competitive firm of Sandmo (1971) within the context of the KMM model. There is one period with two dates, 0 and 1. To begin, the firm produces a single commodity according to a deterministic cost function, \( C(Q) \), where \( Q \geq 0 \) is the output level and \( C(Q) \) is compounded to date 1. The cost function, \( C(Q) \), has the properties that \( C(0) = C'(0) = 0 \), and \( C''(Q) > 0 \) and \( C'''(Q) > 0 \) for all \( Q > 0 \).

At date 1, the firm sells its entire output, \( Q \), at the then prevailing per-unit price, \( \tilde{P} \), which is a positive random variable not known ex ante.\(^9\) The price risk, \( \tilde{P} \), is distributed

\(^8\)The strict convexity of the cost function is driven by the firm’s production technology that exhibits decreasing returns to scale.

\(^9\)Throughout the paper, random variables have a tilde (\( \sim \)) while their realizations do not.
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According to an objective cumulative distribution function (CDF), \( F^\circ(P) \), over support \([P, \overline{P}]\), where \( 0 < P < \overline{P} \). Besides the price risk, \( \tilde{P} \), the firm faces other sources of risk that are aggregated into a single random variable, \( \tilde{Z} \), which is referred to as the background risk and is assumed to be independent of \( \tilde{P} \). The background risk, \( \tilde{Z} \), is additive in nature and is distributed according to an objective CDF, \( G^\circ(Z) \), over support \([Z, \overline{Z}]\), where \( Z < \overline{Z} \).\(^{10}\)

While the background risk, \( \tilde{Z} \), is neither hedgeable nor insurable, the firm can hedge against the price risk, \( \tilde{P} \), by trading infinitely divisible futures and put option contracts at date 0, each of which calls for delivery of one unit of the commodity at date 1.\(^{11}\) The futures price is predetermined at \( P^f \in (P, \overline{P}) \). The put option contracts have a single strike price, \( K \), and an exogenously given option premium, \( \Phi \), per contract, where \( P < K < \overline{P} \) and \( \Phi > 0 \).\(^{12}\) The firm’s profit at date 1 is given by

\[
\Pi(\tilde{P}, \tilde{Z}) = \tilde{P}Q - C(Q) + (P^f - \tilde{P})X + [\max(K - \tilde{P}, 0) - \Phi]Y + \tilde{Z},
\]

where \( X \) is the number of the futures contracts sold (purchased if negative), and \( Y \) is the numbers of the put option contracts purchased (sold if negative) by the firm at date 0. The pair, \((X, Y)\), is referred to as the firm’s hedge position. The firm possesses a von Neumann-Morgenstern utility function, \( u(\Pi) \), defined over its profit at date 1, \( \Pi \), with \( u'(\Pi) > 0 \) and \( u''(\Pi) \leq 0 \). The firm is risk neutral or risk averse, depending on whether \( u(\Pi) = \Pi \) or \( u''(\Pi) < 0 \), respectively.

The firm faces ambiguity in that it is uncertain about the objective CDFs, \( F^\circ(P) \) and \( G^\circ(Z) \). Let \( F(P|\theta) \) and \( G(Z|\theta) \) be the firm’s subjective CDFs of \( \tilde{P} \) and \( \tilde{Z} \), respectively, where \( \theta \) is the realization of an unknown parameter, \( \tilde{\theta} \). The KMM model represents ambiguity by a second-order subjective CDF of \( \tilde{\theta} \), \( H(\theta) \), over support \([\underline{\theta}, \overline{\theta}]\) with \( \underline{\theta} < \overline{\theta} \), which captures the firm’s uncertainty about which of the subjective CDFs govern \( \tilde{P} \) and \( \tilde{Z} \). While

\(^{10}\)The background risk, \( \tilde{Z} \), is allowed to have either a positive, zero, or negative mean.

\(^{11}\)Because of the put-call parity, payoffs of any combinations of futures, calls, and puts can be replicated by any two of these three financial instruments, thereby rendering one of them to be redundant. Restricting the firm to use only futures and put option contracts is without any loss of generality.

\(^{12}\)In practice, the underlying of the futures and put option contracts is at best imperfectly correlated with the price risk, thereby giving rise to basis risk (Briys et al., 1993). Incorporating basis risk into the model complicates the analysis to a great extent. This challenging extension is left for future research.
\( \tilde{P} \) and \( \tilde{Z} \) are first-order independent, they are second-order dependent when the parameter, \( \theta \), is introduced. This can be justified by a learning framework, whereby the first-order priors are the unconditional likelihoods, and the second-order priors are the Bayesian posterior distributions. In this context, \( \theta \) can be interpreted as a latent business cycle indicator that affects the price and background risk simultaneously.\(^{13}\)

The recursive structure of the KMM model implies that the firm’s expected utility under ambiguity can be computed in three steps. First, the firm’s expected utility for each subjective joint CDF of \( \tilde{P} \) and \( \tilde{Z} \) is calculated:

\[
U(\theta) = \int_P \int_Z u[\Pi(P, Z)]dF(P|\theta)dG(Z|\theta),
\]

where \( \Pi(P, Z) \) is given by Eq. (1). Second, each (first-order) expected utility obtained in Eq. (2) is transformed by an increasing function, \( \varphi(U) \), where \( U \) is the firm’s utility level. Finally, the (second-order) expectation of the transformed expected utility obtained in the second step is taken with respect to the second-order subjective CDF of \( \tilde{\theta} \). The firm’s ex-ante decision problem as such is given by

\[
\max_{Q \geq 0, X, Y} \int_{\theta} \varphi[U(\theta)]dH(\theta),
\]

where \( U(\theta) \) is defined by Eq. (2). Inspection of the objective function of program (3) reveals that the effect of ambiguity, represented by the CDF, \( H(\theta) \), and the effect of ambiguity preferences, represented by the shape of the ambiguity function, \( \varphi(U) \), can be separated and thus studied independently.

The firm is said to be ambiguity averse if, for any given triple of output level and hedge position, \((Q, X, Y)\), the objective function of program (3) decreases when the firm’s ambiguous beliefs, specified by \( H(\theta) \), change in a way that induces a mean-preserving spread in the distribution of the firm’s expected utility. According to this definition, Klibanoff et al. (2005) show that ambiguity aversion implies that the ambiguity function, \( \varphi(U) \), is concave

\(^{13}\)The authors would like to thank an anonymous referee for suggesting this interpretation.
in $U$.$^{14}$ To disentangle the effect of ambiguity aversion vis-a-vis that of risk aversion on the firm’s production and hedging decisions, the firm is assumed to be risk neutral, i.e., $u(\Pi) = \Pi$, and ambiguity averse, i.e., $\varphi'(U) > 0$ and $\varphi''(U) < 0$, throughout the paper.

Since $u(\Pi) = \Pi$, the first-order conditions for program (3) are given by

$$
\int_{\theta} \varphi'[U^*(\theta)] [E_F(P|\theta) - C'(Q^*)] dH(\theta) = 0,
$$

(4)

$$
\int_{\theta} \varphi'[U^*(\theta)] [P^I - E_F(\tilde{P}|\theta)] dH(\theta) = 0,
$$

(5)

and

$$
\int_{\theta} \varphi'[U^*(\theta)] \{E_F[\max(K - \tilde{P}, 0)|\theta] - \Phi\} dH(\theta) = 0,
$$

(6)

where $Q^*$ is the optimal output level, $(X^*, Y^*)$ is the optimal hedge position,

$$
U^*(\theta) = E_F(\tilde{P}|\theta)Q^* - C(Q^*) + [P^I - E_F(\tilde{P}|\theta)]X^* + \{E_F[\max(K - \tilde{P}, 0)|\theta] - \Phi\} Y^* + E_G(\tilde{Z}|\theta),
$$

(7)

and $E_F(\cdot|\theta)$ and $E_G(\cdot|\theta)$ are the expectation operators with respect to the marginal CDFs, $F(P|\theta)$ and $G(Z|\theta)$, respectively. The second-order conditions for program (3) are satisfied given the assumed properties of $\varphi(U)$ and $C(Q)$.

**OPTIMAL PRODUCTION DECISION**

Adding Eqs. (4) and (5) yields

$$
\int_{\theta} \varphi'[U^*(\theta)] [P^I - C'(Q^*)] dH(\theta) = 0.
$$

(8)

$^{14}$When $\varphi(U) = -\alpha^{-1}\exp(-\alpha U)$, Klibanoff et al. (2005) show that the maxmin expected utility model of Gilboa and Schmeidler (1989) is the limiting case as the coefficient of absolute ambiguity aversion, $\alpha$, approaches infinity under some conditions.
It then follows from $\varphi'(U) > 0$ and Eq. (8) that $C'(Q^*) = P^f$, thereby invoking the following proposition.

**Proposition 1.** Given that the ambiguity-averse competitive firm can hedge against the ambiguous price risk, $\tilde{P}$, by trading the futures and put option contracts, the firm’s optimal output level, $Q^*$, is the one that equates the marginal cost of production, $C'(Q^*)$, to the predetermined futures price, $P^f$.

The intuition for Proposition 1 is as follows. Since the firm can always sell the last unit of its output through the futures contracts at the predetermined futures price, $P^f$, the usual optimality condition applies in that the marginal cost of production, $C'(Q^*)$, must be equated to the known marginal revenue, $P^f$, which determines the optimal output level, $Q^*$. An immediate implication of Proposition 1 is that the firm’s optimal production decision depends neither on the firm’s attitudes towards ambiguity nor on the incidents to the underlying ambiguity. Proposition 1 as such extends the separation theorem of Danthine (1978), Feder et al. (1980), and Holthausen (1979) to the case of smooth ambiguity preferences and in the presence of ambiguous price and background risk.\(^{15}\)

**OPTIMAL HEDGING DECISION**

To focus on the firm’s pure hedging motive, assume henceforth that the futures and put option contracts are fairly priced in that

$$P^f = \int_P \mathcal{P} \, dF^\circ(P) = \int_0^\mathcal{P} \mathcal{E}_F(\tilde{P} | \theta) dH(\theta),$$

and

$$\Phi = \int_P^K (K - P) dF^\circ(P) = \int_0^\mathcal{P} \mathcal{E}_F[\max(K - \tilde{P}, 0) | \theta] dH(\theta).$$

\(^{15}\)Iwaki and Osaki (2012) and Wong (2015) show that the separation theorem holds under smooth ambiguity preferences when the price risk is the only source of ambiguity.
Eqs. (9) and (10) imply that $P_f$ and $\Phi$ are set equal to the unconditional expected values of $\tilde{P}$ and $\max(K - \tilde{P}, 0)$ with respect to both the objective and subjective CDFs of $\tilde{P}$, respectively. Using Eqs. (9) and (10), Eqs. (5) and (6) can be written as

$$\text{Cov}_H\{\varphi'[U^*(\tilde{\theta})], E_F(\tilde{P}|\tilde{\theta})\} = 0,$$

and

$$\text{Cov}_H\{\varphi'[U^*(\tilde{\theta})], E_F[\max(K - \tilde{P}, 0)|\tilde{\theta}]\} = 0,$$

where $\text{Cov}_H(\cdot, \cdot)$ is the covariance operator with respect to the CDF, $H(\theta)$.

Consider first the case that the expected value of $\tilde{Z}$ is preserved when $\theta$ varies, while that of $\tilde{P}$ is not. When $X^* = Q^*$ and $Y^* = 0$, Eq. (7) implies that $U^*(\theta) = P_f Q^* - C(Q^*) + E_G(\tilde{Z}|\theta)$. The covariance terms in Eqs. (11) and (12) vanish at $X^* = Q^*$ and $Y^* = 0$, thereby invoking the following proposition.

**Proposition 2.** Given that the futures and put option contracts are fairly priced, the ambiguity-averse competitive firm optimally opts for a full-hedge, i.e., $X^* = Q^*$, and uses no options, i.e., $Y^* = 0$, if a change in the parameter, $\theta$, preserves the expected value of the background risk, $\tilde{Z}$, but not that of the price risk, $\tilde{P}$.

The intuition for Proposition 2 is as follows. Since a change in $\theta$ preserves the expected value of $\tilde{Z}$ but not that of $\tilde{P}$, the firm, being risk neutral, regards the background risk as unambiguous while the price risk as ambiguous. Because of ambiguity aversion, the firm is induced to opt for a full-hedge, i.e., $X^* = Q^*$, and use no options, i.e., $Y^* = 0$, in order to completely eliminate any ambiguity arising from the ambiguous price risk. Proposition 2 as such extends the full-hedging theorem to the case of smooth ambiguity preferences and in the presence of ambiguous price and background risk.\textsuperscript{16}

\textsuperscript{16}The results of Proposition 2 are readily extended to the case that $E_G(\tilde{Z}|\theta) = \delta + \gamma E_F(\tilde{P}|\tilde{\theta})$, where $\delta$ and $\gamma$ are constants. In this case, the firm’s optimal hedge position, $(X^*, Y^*)$, satisfies that $X^* = Q^* + \gamma$ and $Y^* = 0$. 
Now consider the case that the expected value of $\tilde{P}$ is preserved when $\theta$ varies, while that of $\tilde{Z}$ is not. Since $E_F(\tilde{P}|\theta) = P^f$ for all $\theta \in [\underline{\theta}, \bar{\theta}]$, the futures contracts are redundant and play no hedging role. Differentiating the objective function of program (3) with respect to $Y$, and evaluating the resulting derivative at $Q = Q^*$ and $Y = 0$ yields

$$\frac{\partial}{\partial Y} \int_\theta^{\bar{\theta}} \varphi[U(\theta)]dH(\theta) \bigg|_{Q=Q^*, Y=0}$$

$$= \int_\theta^{\bar{\theta}} \varphi'[P^f Q^* - C(Q^*) + E_G(\tilde{Z}|\theta)]\{E_F[\max(K - \tilde{P}, 0)|\theta] - \Phi\}dH(\theta)$$

$$= \text{Cov}_H\{\varphi'[P^f Q^* - C(Q^*) + E_G(\tilde{Z}|\theta)], E_F[\max(K - \tilde{P}, 0)|\theta]\}, \quad (13)$$

where the second equality follows from Eq. (10). Since $\varphi''(U) < 0$, the covariance term on the right-hand side of Eq. (13) is positive (negative) if $E_F[\max(K - \tilde{P}, 0)|\theta]$ and $E_G(\tilde{Z}|\theta)$ move in the opposite directions (same direction) when $\theta$ varies. It then follows from Eq. (13) and the second-order conditions for program (3) that $Y^* > (<) 0$, thereby invoking the following proposition.

**Proposition 3.** Given that the futures and put option contracts are fairly priced, and that a change in the parameter, $\theta$, preserves the expected value of the price risk, $\tilde{P}$, but not that of the background risk, $\tilde{Z}$, the ambiguity-averse competitive firm optimally opts for a long (short) put option position, i.e., $Y^* > (<) 0$, if $E_F[\max(K - \tilde{P}, 0)|\theta]$ and $E_G(\tilde{Z}|\theta)$ move in the opposite directions (same direction) when $\theta$ varies.

The intuition for Proposition 3 is as follows. Since a change in $\theta$ preserves the expected value of $\tilde{P}$ but not that of $\tilde{Z}$, the firm, being risk neutral, regards the price risk as unambiguous while the background risk as ambiguous. Since $E_F(\tilde{P}|\theta) = P^f$ for all $\theta \in [\underline{\theta}, \bar{\theta}]$, the futures contracts are redundant and play no hedging role. However, the put option contracts are not redundant and serve as a useful hedging instrument to hedge against the ambiguous
background risk.\textsuperscript{17} To see this, differentiating $U^*(\theta)$ with respect to $E_F[\max(K - \tilde{P}, 0)|\theta]$ yields

$$\frac{\partial U^*(\theta)}{\partial E_F[\max(K - \tilde{P}, 0)|\theta]} = Y^* + \frac{\partial E_G(\tilde{Z}|\theta)}{\partial E_F[\max(K - \tilde{P}, 0)|\theta]},$$

where $U^*(\theta)$ is given by Eq. (7) with $E_F(\tilde{P}|\theta) = P^f$. Eq. (12) implies that the optimal put option position, $Y^*$, is the one that minimizes the variability of $\varphi'[U^*(\theta)]$ with respect to changes in $E_F[\max(K - \tilde{P}, 0)|\theta]$. It then follows from Eq. (14) that this goal is achieved by setting $Y^* > (<) 0$ if $\partial E_G(\tilde{Z}|\theta)/\partial E_F[\max(K - \tilde{P}, 0)|\theta] < (>) 0$.

**PARAMETRIC EXAMPLES**

In the general case that neither the expected value of $\tilde{P}$ nor that of $\tilde{Z}$ is preserved when $\theta$ varies, the firm regards the price and background risk as ambiguous. To gain more insight into this general case, suppose that the parameter, $\tilde{\theta}$, is uniformly distributed over support $[-\hat{\theta}, \hat{\theta}]$, where $\hat{\theta} > 0$. The price risk, $\tilde{P}$, is uniformly distributed over support $[P + \theta, \bar{P} + \theta]$, where $\bar{P} > \hat{\theta}$. The background risk, $\tilde{Z}$, is uniformly distributed over support $[-\hat{Z} + \gamma \theta^2, \hat{Z} + \gamma \theta^2]$, where $\hat{Z} > 0$. The firm’s output level is set equal to unity and the production cost is normalized to zero. In this example, Eq. (7) becomes

$$U^*(\theta) = \theta + \frac{P + \bar{P}}{2} - \theta X^* + \left[\frac{(K - P - \theta)^2}{2(\bar{P} - P)} - \frac{(K - P + \hat{\theta})^3 - (K - P - \hat{\theta})^3}{12 \hat{\theta} (\bar{P} - P)}\right] Y^* + \gamma \theta^2,$$

where $K > P + \hat{\theta}$. It is evident from Eq. (15) that $U^*(\theta)$ is invariant to changes in $\theta$ if $X^* = 1 + 2\gamma(K - P)$ and $Y^* = -2\gamma(\bar{P} - P)$, thereby making $(X^*, Y^*)$ the solution to Eqs. (5) and (6).

\textsuperscript{17}Suppose that $\tilde{P}$ is uniformly distributed over the support $[P - \theta, \bar{P} + \theta]$, where $\theta < P$. Hence, $E_F(\tilde{P}|\theta) = (P + \bar{P})/2$, which is invariant to changes in $\theta$. On the other hand, $E_F[\max(K - P, 0)|\theta] = (K - P + \theta)^2/2(\bar{P} - P + 2\theta)$, which is increasing in $\theta$. 

In the above example, hedging is perfect in the sense that the optimal hedge position, $(X^*, Y^*)$, completely eliminates all ambiguity. Suppose now that the background risk, $\tilde{Z}$, is uniformly distributed over support $[-\tilde{Z} + \gamma \theta^3, \tilde{Z} + \gamma \theta^3]$ so that perfect hedging is not admissible. To numerically solve the optimal hedge position, consider the case that the firm’s preferences exhibit constant absolute ambiguity aversion, i.e., $\varphi(U) = -\alpha^{-1} \exp(-\alpha U)$, where $\alpha > 0$ is the coefficient of absolute ambiguity aversion. The parameter values are chosen as follows: $P = 1$, $\bar{P} = 5$, $K = 3$, $\hat{\theta} = 1$, and $\gamma = 1$. Figure 1 depicts the optimal hedge positions for different coefficients of absolute ambiguity aversion.

![Figure 1](image-url)

**FIGURE 1**

Optimal hedge positions for different coefficients of absolute ambiguity aversion

As is evident from Figure 1, the optimal futures position, $X^*$, decreases as the coefficient of absolute ambiguity aversion, $\alpha$, increases, i.e., as the firm becomes more ambiguity averse. $X^*$ is an over-hedge, i.e., $X^* > 1$, or an under-hedge, i.e., $X^* < 1$, depending on whether
α is smaller or greater than 7, respectively. On the other hand, the optimal put option position, \( Y^* \), is a long position that increases with an increase in α. This is driven by the fact that greater ambiguity aversion induces the firm to hedge more against the non-linear source of ambiguity embedded in the background risk. Since the long put option position also reduces the linear source of ambiguity embedded in the price risk, the firm optimally reduces its short futures position. As such, the futures and put option contracts act like substitutes for each other.

**CONCLUSION**

This paper examines the production and hedging decisions of the competitive firm under ambiguous price and background risk. The firm’s preferences exhibit smooth ambiguity aversion developed by Klibanoff et al. (2005). Within the KMM model, ambiguity is represented by a second-order probability distribution that captures the firm’s uncertainty about which of the subjective beliefs govern the price and background risk. On the other hand, ambiguity preferences are modeled by the (second-order) expectation of a concave transformation of the (first-order) expected utility of profit conditional on each plausible subjective joint distribution of the price and background risk.

The firm’s optimal production decision is shown to be independent of the firm’s smooth ambiguity preferences and of the underlying ambiguous price and background risk, thereby invoking the separation theorem. Given that the firm can trade fairly priced futures and option contracts for hedging purposes, the full-hedging theorem holds only when the background risk is unambiguous. When the price risk is unambiguous, the unbiased futures contracts become redundant but the option contracts are not due to the prevalence of ambiguity driven by the volatility of the price risk. Hence, the option contracts serve as a useful hedging instrument to hedge against the ambiguous background risk. In the general case that both the price risk and the background risk are ambiguous, examples are constructed to show that the firm indeed includes the option contracts in its optimal hedge
position. This paper as such provides a rationale for the hedging role of options under smooth ambiguity preferences and ambiguous price and background risk.

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